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Dimensional reduction of the Kähler–Yang–Mills equations

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Abstract

This master thesis studies dimensional reductions of the Kähler–Yang–Mills equations over compact Kähler manifolds by imposing some type of $SU(2)$-equivariance, to obtain the *gravitational vortex equations*. When restricted to compact Riemann surfaces it is possible to explore some conditions for the existence of solutions based on the stability of holomorphic triples.

Keywords: *Kähler–Yang–Mills equations, moduli space, vector bundle, Hitchin–Kobayashi correspondence, holomorphic triples.*

Resumen

Este trabajo de fin de máster se centra en el estudio de reducciones dimensionales de las ecuaciones de Kähler–Yang–Mills sobre variedades de Kähler compactas al imponer cierto tipo de equivariancia bajo la acción de $SU(2)$, para obtener las ecuaciones de vórtices gravitacionales. Al restringirse a superficies de Riemann compactas, es posible explorar condiciones para la existencia de soluciones basadas en una noción de estabilidad de ternas holomorfas.

Palabras clave: *ecuaciones de Kähler–Yang–Mills , espacio de móduli, fibrado vectorial, correspondencia de Hitchin–Kobayashi, ternas holomorfas.*

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Contents

Introduction

This master thesis is devoted to the study of dimensional reductions of the Kähler–Yang–Mills equations. These are a recent addition to the class of gauge-theoretic equations in differential geometry introduced by Álvarez-Cónsul, García-Fernández and García-Prada (see [1] and [13]). We will be mostly concerned about compact Kähler manifolds and holomorphic vector bundles over them. We now briefly recall some historical facts about the advancements in this area of research.

Vector bundles are central objects in both algebraic and differential geometry. A first result concerning the classification of vector bundles of rank 1 over algebraic varieties is the Abel–Jacobi theorem, although stated in the equivalent language of divisors. A topological classification of complex line bundles is entirely possible in terms of the first Chern class. However, this fails for higher ranks. A relevant result which solves this problem in a particular case is Grothendieck's decomposition theorem: every holomorphic vector bundle over the complex projective line $\mathbb{P}^1$ is equivalent to a direct sum of line bundles, all of different degree (first Chern class). Atiyah [5] extended the classification to genus 1 (algebraic elliptic curves).

A new tool for studying vector bundles developed in the 1960's with the appearance of Mumford's Geometric Invariant Theory and the notion of stability [26]. This allowed for the construction of quotients in the context of algebraic geometry, and this is convenient for the study of vector bundles as we are often interested in equivalence classes of these objects. With this concept of stability, Narasimhan and Seshadri proved in 1965 [27] a remarkable theorem, stating that stable holomorphic vector bundles over a Riemann surface are in one-to-one correspondence with projective unitary representations of the fundamental group of the surface (nowadays called $PSU(n)$-character variety).

A few years later, Atiyah and Bott introduced novel ideas from Yang–Mills theory, originally a field-theoretic model in particle physics, to algebraic and differential geometry [6]. Building on these ideas, Donaldson gave a new proof of the Narasimhan–Seshadri theorem in gauge-theoretic terms: an indecomposable vector bundle over a Riemann surface is stable if and only if it admits a projectively flat unitary connection.

The Hermitian–Yang–Mills equations provide a generalization of the projectively flat condition for vector bundles on higher dimensional Kähler manifolds. These equations were already being studied by Kobayashi and Lübke, who independently proved in 1983 [24] that holomorphic vector bundles over compact Kähler manifolds of arbitrary dimension admitting Hermitian–Yang–Mills connections are necessarily stable in the sense of Mumford. What today is known as the Hitchin–Kobayashi correspondence was established as a conjecture in the late 70s independently by Hitchin [18] and Kobayashi [21]. What Kobayashi and Lübke proved was one of the implications of the correspondence.

In 1985, Donaldson proved a partial converse for the case of algebraic surfaces [9]. Shortly after, Uhlenbeck and Yau gave a proof for general compact Kähler manifolds of arbitrary dimension [32], followed by a proof by Donaldson for projective algebraic manifolds [10].

Another relevant equation is that of having constant scalar curvature for a Kähler metric on a compact complex manifold. This is a fourth order nonlinear partial differential equation. Fujiki [12] first gave an interpretation of the Riemannian constant scalar curvature condition in terms of a symplectic moment map, and Donaldson generalized it [11] to the Hermitian scalar curvature in almost Kähler manifolds. These results were already known to Quillen in the case of Riemann surfaces.

The **Kähler–Yang–Mills** equations are related to both the Hermitian–Yang–Mills and constant scalar curvature equations. Consider a compact complex manifold M and a holomorphic vector bundle E over M. The Kähler–Yang–Mills equations for a Hermitian metric h on E and Kähler metric ω on M read as

$$\begin{aligned} i\Lambda_\omega F_h &= \lambda \mathrm{Id}_E, \\ S_\omega - \alpha \Lambda_\omega \operatorname{tr} F_h \wedge F_h &= c, \end{aligned}$$

for constants λ, c. These were introduced in García-Fernández's PhD thesis [13] (see also [1]). Here α is called the coupling parameter. A relevant insight is that these equations also appear as the vanishing moment map condition for a symplectic action on a particular moduli space. It was shown by Álvarez-Cónsul, García-Fernández and García-Prada [1] that bundles admitting Hermitian–Yang–Mills metrics over manifolds M with finite automorphism group and a constant scalar curvature Kähler metric admit solutions to the Kähler–Yang–Mills equations by means of a deformation procedure, for small values of the coupling parameter.

The present work studies dimensional reductions of the Kähler–Yang–Mills equations. More concretely, we assume the base space to be of the form $M = X \times \mathbb{P}^1$, where $\mathbb{P}^1$ denotes the complex projective line (the Riemann sphere), which is isomorphic to $SU(2)/U(1)$, therefore allowing a natural action of $SU(2)$. We assume a particular $SU(2)$-invariant structure on the vector bundle E over $X \times \mathbb{P}^1$ involving two vector bundles E_1, E_2 and a map ϕ between them, a structure called holomorphic triple. This setting allows for a search of solutions to the Kähler–Yang–Mills equations which are invariant under this $SU(2)$-action. This work arrives at what we call the **gravitational vortex equations**.

Álvarez-Cónsul, García-Fernández and García-Prada had already studied this type of dimensional reduction of the Kähler–Yang–Mills equations by considering line bundles over Riemann surfaces, obtaining the Abelian gravitational vortex equation. This builds upon a series of results concerning the vortex equations, studied by García-Prada in [15]. The vortex equations had been introduced by Landau and Ginzburg [23] while studying superconductivity and they were obtained by Jaffe and Taubes [20] for $\mathbb{R}^2$ and by Witten for the hyperbolic plane through a similar process of dimensional reduction from the Yang–Mills equations.

In this work we focus more narrowly on the gravitational vortex equations on Riemann surfaces with holomorphic triples (E_1, E_2, ϕ) of higher rank. In that case, the gravitational vortex equations for two Hermitian metrics h_1,h_2 and a Kähler metric ω read as

$$\begin{cases} i\Lambda_\omega F_{h_1} + \frac{1}{4}\phi \circ \phi^* = 2\pi\tau \mathrm{Id}_{E_1}, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}\phi^* \circ \phi = 2\pi\tau' \mathrm{Id}_{E_2}, \\ S_\omega - \frac{16\pi i\alpha}{\sigma}\Lambda_\omega \operatorname{tr} F_{h_2} \\ \qquad -\alpha i \Lambda_\omega \left[\operatorname{tr}(F_{h_1} \circ \phi \circ \phi^*) - \operatorname{tr}(F_{h_2} \circ \phi^* \circ \phi) + \operatorname{tr}(\partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*)\right] = C, \end{cases}$$

in terms of some constant C, the coupling parameter α and some symmetry-breaking parameters. Holomorphic triples turn out to be key in the study of our equations, as had already been shown by Bradlow and García-Prada [7] and together with Gothen [8] while studying the vortex equations on triples. A notion of stability for triples characterizes the existence of solutions to the vortex equations, and being the latter a subset of our gravitational vortex equations they provide necessary conditions for the existence of solutions. Previous results [7, 8] showed that the moduli spaces of stable triples over compact Riemann surfaces are quasi-projective algebraic varieties and computed the dimension, helping understand the space of solutions to the coupled vortex equations. Fixing arbitrary ranks and degrees of the triples, Pasotti and Prantil studied these moduli spaces on surfaces of genus one [30] and zero [29], and in this direction, To's doctoral dissertation [31] gave specific examples of these moduli spaces over the Riemann sphere.

Inspired by the existence results for the Abelian gravitational vortex equations involving a notion of GIT stability on the space of triples [2, 4, 14], this work proposes a conjecture characterizing the existence of solution to the gravitational vortex equations for genus zero.

Conjecture. *Let $\mathbb{P}^1$ be the projective line and $T = (E_1, E_2, \phi)$ a holomorphic triple over $\mathbb{P}^1$. The following are equivalent.*

1. *There exists a solution to the gravitational vortex equations on $(\mathbb{P}^1, T)$.*
2. *The triple T is τ-polystable and its equivalence class is furthermore GIT-polystable for the action of $SL(2, \mathbb{C})$ on the moduli space of stable triples $\mathcal{M}$.*

This work constitutes a first step to the study of the gravitational vortex equations, taking the initial steps towards a deeper understanding of the dimensional reductions of the Kähler–Yang–Mills equations. Such a project will be the main topic of the author's prospective PhD dissertation.

Outline

Chapter 1 introduces the Kähler–Yang–Mills equations describing in some detail the interpretation of these in terms of a moment map. Several concepts related to standard gauge theory are introduced, such as the space of connections, the gauge group and the corresponding moduli space, together with some links between connections and holomorphic structures on vector bundles.

Chapter 2 contains the main result of this work, namely a dimensional reduction that generalizes that of [2], extending to the case of arbitrary dimension of the base space and arbitrary ranks of two vector bundles over the base space. The resulting equations are coined *gravitational vortex equations*. These equations can be then reduced to simpler situations. In particular, we consider the case of line bundles over arbitrary compact Kähler manifolds and vector bundles of arbitrary rank over Riemann surfaces. The combination of both situations yields the Abelian gravitational vortex equations, for which we recall previous existence results.

Chapter 3 introduces the necessary notions to understand the existence results such as the Hitchin–Kobayashi correspondence and Mumford's stability of vector bundles. We introduce the notion of holomorphic triple and stability, recalling prior results for moduli spaces. Section 3.4 states a conjecture regarding the existence of solutions of the gravitational vortex equations over the complex projective line and higher rank vector bundles in terms of a notion of GIT stability, which is explained at the beginning of the section. This conjecture will become a primary objective of the author's doctoral dissertation. We end the document by recalling some examples of moduli spaces of stable triples over the projective line in order to give a precise feeling of how this objects look like and how a GIT action might be implemented.

Chapter 1

Kähler–Yang–Mills equations

This chapter introduces the Kähler–Yang–Mills equations, the central object of study of this work. These equations were introduced by Álvarez-Cónsul, García-Fernández and García-Prada in [1], and have been studied for some time. The **Kähler–Yang–Mills** equations couple a Kähler metric on a compact complex manifold with a Hermitian metric on a complex vector bundle over the complex manifold. These equations they can be formulated in greater generality in principal bundles, although this work focuses on the linear version. The Kähler–Yang–Mills equations are related to the Hermitian–Yang–Mills and the constant scalar curvature Kähler metric equations. We will often refer to the equations simply as KYM.

Let M be a compact complex manifold and consider a holomorphic vector bundle E over M of rank r. In order to write the equations we introduce some notation which will be mainly following Atiyah–Bott [6]. Let H be an arbitrary Hermitian metric on E. Consider the Killing form

$$\mathrm{tr} : \mathfrak{u}(r) \times \mathfrak{u}(r) \to \mathbb{R}.$$

This is a symmetric bilinear form which is furthermore invariant under the adjoint action of $U(r)$ (actually under the adjoint action of the full $GL(r, \mathbb{C})$), and therefore it defines a bilinear pairing on $\mathrm{End}E$. This can be naturally extended to give a pairing on the space of $\mathrm{End}E$-valued differential forms

$$\Omega^p(\mathrm{End}E) \times \Omega^q(\mathrm{End}E) \to \Omega^{p+q}$$

which will be denoted by $\mathrm{tr}\, a_p \wedge a_q$ for $a_p \in \Omega^p(\mathrm{End}E)$ and $a_q \in \Omega^q(\mathrm{End}E)$. The Kähler–Yang–Mills equations for a Kähler metric (we will abuse notation and refer to the associated Kähler form ω instead) and a Hermitian metric H on E read as

$$\begin{aligned} i\Lambda_\omega F_H &= \lambda \mathrm{Id}_E, \\ S_\omega - \alpha \Lambda_\omega^2 \,\mathrm{tr}\, F_H \wedge F_H &= c. \end{aligned} \tag{1.1}$$

Here, Λ_ω is the adjoint of the Lefschetz operator $L : \eta \mapsto \omega \wedge \eta$, which coincides with the contraction $\Lambda_\omega \eta = \eta \lrcorner \omega^\#$, where $\omega^\#$ is the contravariant counterpart of ω induced by symplectic duality. F_H is the curvature of the Chern connection determined by the Hermitian metric H and the holomorphic structure of E, S_ω is the scalar curvature of the Kähler metric and λ and c are two constants that are completely determined by the topology of E and the cohomology class determined by ω. We refer to [19, 22] as basic sources for the concepts of complex geometry appearing here. The first line in (1.1) constitutes the Hermite–Einstein or Hermitian–Yang–Mills equations.

It is worth noting that for Riemann surfaces $M = \Sigma$ the two equations decouple due to the vanishing of the quadratic term $F_H \wedge F_H$, and the problem reduces to a combination of the

Uniformization Theorem for Riemann surfaces and the Narasimhan–Seshadri theorem. The study of the KYM equations is therefore more interesting in the higher-dimensional case. In [2] the authors analyzed the case when M is the product of a Riemann surface with the complex projective line $\Sigma \times \mathbb{P}^1$, and a particular rank-two holomorphic vector bundle which is equivariant under a natural $SU(2)$-action. This allowed for the search of $SU(2)$-invariant solutions of the KYM equations, and this was equivalent to finding solutions of the so-called *gravitational vortex equations* over the Riemann surface and a line bundle over it.

1.1 Moment map and the Hermitian–Yang–Mills equations

Both the constant scalar curvature Kähler and the Hermitian–Yang–Mills equations have been interpreted in terms of vanishing moment map conditions. This notion arises in symplectic geometry and some of its features will be outlined here, but we refer to [25] for details. Throughout the next sections we will introduce moment map interpretations for the Hermitian–Yang–Mills and constant scalar curvature Kähler metric in order to construct a moment map for the Kähler–Yang–Mills equations.

Consider a compact Lie group G with Lie algebra $\mathfrak{g}$. Let G act by symplectomorphisms on a symplectic manifold (M, ω). There is an induced Lie algebra homomorphism $\mathfrak{g} \to \mathcal{X}(M)$, mapping each element $\xi \in \mathfrak{g}$ to the vector field defined by $X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(t\xi)\cdot p$. The action is called Hamiltonian if each such vector field X_ξ is Hamiltonian, i.e. if to each $\xi \in \mathfrak{g}$ we can assign a differentiable function H_ξ and furthermore this assignment is a Lie algebra homomorphism, where $C^\infty(M)$ is endowed with the Poisson bracket with respect to the symplectic form ω. In this context, a **moment map** for the action is a differentiable map $\mu : M \to \mathfrak{g}^*$ satisfying

$$H_\xi(p) = \langle \mu(p), \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between the Lie algebra and its dual $\mathfrak{g}^*$, and that is furthemore equivariant under the coadjoint action of G on $\mathfrak{g}^*$.

Moment maps provide a way of constructing symplectic manifolds from a given one endowed with a Hamiltonian G-action. We consider the level set $\mu^{-1}(0) \subset M$. Due to the equivariance of the moment map, $\mu^{-1}(0)$ is invariant under the action of G, since $0 \in \mathfrak{g}^*$ is certainly a fixed point of the coadjoint action. If 0 is a regular value of the moment map, and G acts freely and properly on $\mu^{-1}(0)$ then not only is $\mu^{-1}(0)/G$ a manifold but it admits a natural symplectic structure. The quotient has dimension $\dim(\mu^{-1}(0)/G) = \dim M - 2\dim G$. This is often called the Marsden–Weinstein quotient. Further details of this construction can be found in Section 5.4. of [25].

Fujiki [12] and Donaldson [11] interpreted the scalar curvature equation in terms of a moment map. This was also known to Quillen in the case of Riemann surfaces. The Hermitian–Yang–Mills equations were also interpreted in these symplectic terms first by Atiyah and Bott [6] over Riemann surfaces and by Donaldson [9] in higher dimensions. We will briefly recall the basics of these constructions and provide a moment map interpretation for the KYM equations.

We start by giving a moment map interpretation of the Hermitian–Yang–Mills equations. Let (M, ω) be a symplectic manifold. Let E be a differentiable complex vector bundle over M together with an Hermitian metric H, with $\operatorname{rank} E = r$. We consider the space $\mathcal{A}$ of unitary connections on E, which is an affine space modelled on $\Omega^1(\mathrm{ad}E_H)$. Here E_H denotes the principal bundle of unitary frames associated to the Hermitian vector bundle (E, H) and $\mathrm{ad}E_H$ is the adjoint bundle, to be identified with skew-Hermitian endomorphisms of (E, H).

Let $\mathcal{G}$ be the unitary gauge group of E_H, i.e. the set of $U(r)$-equivariant diffeomorphisms of E_H covering the identity over M. The unitary gauge group acts upon the space of connections from the left in a natural manner. One way to state this action requires seeing the connections as $U(r)$-equivariant splittings of the sequence

$$0 \to VE_H \hookrightarrow TE_H \to \pi^*TM \to 0,$$

i.e. as $U(r)$-equivariant mappings $A : TE_H \to VE_H$. The action of the gauge group is from the left via the pushforward $g \cdot A := g_* \circ A \circ g_*^{-1}$. We can provide $\mathcal{A}$ with a symplectic form defined by

$$\omega_{\mathcal{A}}(a, b) = \int_M \mathrm{tr}\, a \wedge b \wedge \frac{\omega^{n-1}}{(n-1)!}$$

for $a, b \in T_A\mathcal{A} \simeq \Omega^1(\mathrm{ad}E_H)$, where $\mathrm{tr} \cdot \wedge \cdot$ denotes the pairing $\Omega^p(\mathrm{ad}E_H) \times \Omega^q(\mathrm{ad}E_H) \to \Omega^{p+q}$. This 2-form is a symplectic form and the main result to be recalled is that the gauge group acts in a Hamiltonian fashion on $\mathcal{A}$, the moment map being given by $\mu_{\mathcal{G}} : \mathcal{A} \to \mathrm{Lie}\,\mathcal{G}^*$

$$\langle \mu_{\mathcal{G}}(A), \zeta \rangle = \int_M \mathrm{tr}\, \zeta \wedge (i\Lambda_\omega F_A - \lambda \mathrm{Id}_E)\frac{\omega^n}{n!}, \tag{1.2}$$

for $\zeta \in \mathrm{Lie}\,\mathcal{G} \simeq \Omega^0(\mathrm{ad}_H E)$ and for a fixed but arbitrary $\lambda \in \mathbb{R}$.

If the base space M is furthermore a Kähler manifold there is a distinguished subspace of connections $\mathcal{A}_J^{1,1} \subset \mathcal{A}$ satisfying $F_A \in \Omega^{1,1}(\mathrm{ad}E_H)$ where the complex structure J in M allows for the (p, q)-type decomposition. In this context, a connection A is called Hermitian–Yang–Mills if it satisfies

$$i\Lambda_\omega F_A = \lambda \mathrm{Id}_E,$$

and it is easy to prove that λ depends only on the cohomology class $[\omega]$ and on the topology of the bundle:

$$\deg_\omega E = \frac{i}{2\pi}\int_M \mathrm{tr}\,\Lambda_\omega F_A \frac{\omega^n}{n!} = \frac{1}{2\pi}\int_M \lambda\, \mathrm{tr}\,\mathrm{Id}_E = \lambda \frac{\mathrm{rank}\, E\, \mathrm{Vol}_\omega M}{2\pi}.$$

The vanishing locus of the moment map corresponds precisely to the set of solutions to the Hermitian–Yang–Mills equations. The moduli space $\mu_{\mathcal{G}}^{-1}(0)/\mathcal{G}$ of gauge-equivalent solutions is then an infinite-dimensional version of a Marsden–Weinstein quotient. The existence of a complex structure on $\mathcal{A}_J^{1,1}$ away from its singularities then implies that the moduli space of Hermitian–Yang–Mills connections is (away from singularities) a Kähler quotient, once it is shown that this complex structure is compatible with $\omega_{\mathcal{A}}$.

1.2 Moment map and constant scalar curvature

Let (M, ω) be a compact symplectic manifold and let $\mathcal{J}$ be the space of almost complex structures compatible with ω. Let $\mathcal{H}$ be the group of Hamiltonian symplectomorphisms of (M, ω). We can define a symplectic form for $\mathcal{J}$ in the following way: let $J \in \mathcal{J}$. The tangent space $T_J\mathcal{J}$ is composed of vector bundle endomorphisms $\Phi : TM \to TM$ symmetric with respect to the metric $g_J = \omega(\cdot, J\cdot)$ satisfying $\Phi \circ J = -J \circ \Phi$. For such $\Phi, \Psi \in T_J\mathcal{J}$ we define

$$\omega_{\mathcal{J}}(\Phi, \Psi) = \frac{1}{2}\int_M \mathrm{tr}(J \circ \Phi \circ \Psi)\frac{\omega^n}{n!}$$

This can be shown to be a symplectic 2-form and the group of Hamiltonian symplectomorphisms acts upon $\mathcal{J}$ from the left in a Hamiltonian fashion via $h \cdot J = h_* \circ J \circ h_*^{-1}$. The space $\mathcal{J}$ admits itself a complex structure given by $\Phi \in T_J\mathcal{J} \mapsto J \circ \Phi \in T_J\mathcal{J}$, which is furthermore compatible with $\omega_{\mathcal{J}}$, turning $\mathcal{J}$ into an infinite-dimensional Kähler manifold.

Any almost complex structure $J \in \mathcal{J}$ compatible with ω induces a Hermitian metric on T^*M. There is a unique unitary connection on T^*M whose $(0,1)$-component coincides with the standard Dolbeault operator $\overline{\partial}_J : \Omega_J^{p,q} \to \Omega_J^{p,q+1}$. The Hermitian scalar curvature S_J is the real function on M defined by

$$S_J \frac{\omega^n}{n!} = 2\rho_J \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Here ρ_J is the real 2-form defined as $-i$ times the induced curvature in the determinant bundle $\Lambda^n T^*M$. This S_J is defined so that for integrable almost complex structures J it coincides with the Riemannian scalar curvature induced by the Riemannian metric $\omega(\cdot, J\cdot)$. Donaldson proved [11] that the action of $\mathcal{H}$ on $\mathcal{J}$ is Hamiltonian with moment map given by

$$\langle \mu_{\mathcal{H}}(J), \eta_\phi \rangle = -\int_M \phi S_J \frac{\omega^n}{n!},$$

where $\eta_\phi \in \operatorname{Lie}\mathcal{H}$ is the Hamiltonian vector field generated by $\phi \in C_0^\infty(M)$, the zero-average Hamiltonian functions that determines the field. The vanishing locus of the moment map coincides precisely with the almost complex structures J such that the scalar curvature is constant $S_J = c$.

1.3 Moment map intrepretation of the Kähler–Yang–Mills equations

Consider a compact symplectic manifold (M, ω) and (E, H) a Hermitian vector bundle over M. It is possible to extend the gauge group when the base space is endowed with a symplectic structure. This extension of Lie groups is given by

$$1 \to \mathcal{G} \overset{i}{\hookrightarrow} \tilde{\mathcal{G}} \overset{p}{\twoheadrightarrow} \mathcal{H} \to 1. \tag{1.3}$$

Here $\tilde{\mathcal{G}}$ is defined to be the set of $U(r)$-equivariant diffeomorphisms of E_H covering Hamiltonian symplectomorphisms of the base space, and i is the natural inclusion $\mathcal{G} \subset \tilde{\mathcal{G}}$. It can be shown that p is surjective by taking a unitary connection on E and horizontally lifting a Hamiltonian vector field; the flow that it generates will give a suitable element of $\tilde{\mathcal{G}}$.

Consider now the affine space $\mathcal{A}$ of connections on E and $\mathcal{J}$ the space of almost complex structures compatible with ω. We now define a family of symplectic structures in $\mathcal{J} \times \mathcal{A}$ parametrized by $\alpha \in \mathbb{R}$ as

$$\omega_\alpha = \omega_{\mathcal{J}} + 4\alpha\omega_{\mathcal{A}}.$$

The extended gauge group $\tilde{\mathcal{G}}$ acts upon $\mathcal{J} \times \mathcal{A}$ from the left via

$$g \cdot (J, A) = (p(g) \cdot J, g \cdot A)$$

where $p(g) \in \mathcal{H}$ acts upon J as defined above. The main result to be recalled in this section from [1] is that this action is Hamiltonian with respect to the symplectic form ω_α. The definition of the moment map requires the construction of an assignment of a linear map $\theta_A : \operatorname{Lie}(\operatorname{Aut} E_H) \to$

$\mathrm{Lie}\,\mathcal{G}$ to each connection $A \in \mathcal{A}$ yielding a linear splitting of the so-called Atiyah short exact sequence

$$0 \to \mathrm{Lie}\,\mathcal{G} \to \mathrm{Lie}(\mathrm{Aut}E_H) \to \mathrm{Lie}(\mathrm{Diff}M) \to 0.$$

For $\zeta \in \mathrm{Lie}\,\tilde{\mathcal{G}}$, $\theta_A(\zeta)$ is an element of $\mathrm{Lie}\,\mathcal{G} \simeq \Omega^0(\mathrm{ad}E_H)$ and $p(\zeta) = \eta_\phi$ is an element of $\mathrm{Lie}\,\mathcal{H} \simeq C_0^\infty(M)$, where ϕ is the Hamiltonian function generating the vector field η_ϕ. There is yet another map $\theta_A^\perp : \mathrm{Lie}\,\mathcal{H} \to \tilde{\mathcal{G}}$ uniquely defined by $\mathrm{Id}_{\tilde{\mathcal{G}}} = i \circ \theta_A + \theta_A^\perp \circ p$. In this context, the equivariant moment map is given by its pairing with an arbitrary element $\zeta \in \tilde{\mathcal{G}}$:

$$\begin{aligned}\langle \mu_\alpha(J,A), \zeta\rangle = &-4\alpha \int_M \mathrm{tr}\,\theta_A(\zeta) \wedge (i\Lambda_\omega F_A - \lambda \mathrm{Id}_E)\frac{\omega^n}{n!} \\ &- \int_M \phi \left\{ S_J - \alpha\Lambda_\omega^2 \,\mathrm{tr}\, F_A \wedge F_A + 4\alpha\Lambda_\omega \,\mathrm{tr}\, F_A \wedge \lambda \mathrm{Id}_E \right\} \frac{\omega^n}{n!}.\end{aligned}$$

There is a formally integrable complex structure on $\mathcal{J} \times \mathcal{A}$ given by

$$I_{J,A}(J', a) = (J \circ J', -a \circ J)$$

for $J' \in T_J\mathcal{J}$, $a \in T_A\mathcal{A} \simeq \Omega^1(\mathrm{ad}E_H)$, and for positive α, it is compatible with the family of symplectic structures given above. Now we define the subspace $\mathcal{P} \subset \mathcal{J} \times \mathcal{A}$ consisting of pairs (J, A) where J is an integrable almost complex structure and A is a connection on E with curvature of type $(1,1)$ with respect to J. This is a $\tilde{\mathcal{G}}$-invariant and Kählerian subspace. A pair $(J, A) \in \mathcal{P}$ is said to satisfy the Kähler–Yang–Mills equations if the following hold:

$$\begin{aligned} i\Lambda_\omega F_A &= \lambda \mathrm{Id}_E, \\ S_J - \alpha\Lambda_\omega^2 \,\mathrm{tr}\, F_A \wedge F_A &= c. \end{aligned}$$

Here $\lambda, c \in \mathbb{R}$. Now we can prove that the vanishing locus of the moment map (restricted to $\mathcal{P}$) coincides with the set of solutions of the Kähler–Yang–Mills equations. Assume $(J, A) \in \mu_\alpha^{-1}(0) \cap \mathcal{P}$. Take any arbitrary element $\eta \in \mathcal{H}$ and consider $\theta_A^\perp(\eta) \in \mathrm{Lie}\,\tilde{\mathcal{G}}$. Pair this element with $\mu_\alpha(J, A) = 0$ to get

$$\begin{aligned} 0 = &-4\alpha \int_M \mathrm{tr}\,\theta_A\theta_A^\perp\eta \wedge (i\Lambda_\omega F_A - \lambda\mathrm{Id}_E)\frac{\omega^n}{n!} \\ &- \int_M \phi(S_J - \alpha\Lambda_\omega^2 \,\mathrm{tr}\, F_A \wedge F_A + 4\alpha\Lambda_\omega \,\mathrm{tr}\, F_A \wedge \lambda\mathrm{Id}_E)\frac{\omega^n}{n!} \end{aligned}$$

where $p(\theta_A^\perp\eta) = \eta_\phi \in \mathcal{H}$. This implies in particular that the term in brackets within the second integral must be constant

$$S_J - \alpha\Lambda_\omega^2 \,\mathrm{tr}\, F_A \wedge F_A + 4\alpha\Lambda_\omega \,\mathrm{tr}\, F_A \wedge \lambda\mathrm{Id}_E = c',$$

but evaluating again at an arbitrary element $\zeta \in \mathrm{Lie}\,\tilde{\mathcal{G}}$ we conclude that $i\Lambda_\omega F_A = \lambda\mathrm{Id}_E$, so in particular

$$S_J - \alpha\Lambda_\omega^2 \,\mathrm{tr}\, F_A \wedge F_A = c' - 4\alpha|\lambda|^2 = c.$$

This shows that the pair (J, A) satisfies the Kähler–Yang–Mills equations. The converse easily follows from the expression for the moment map. In Chapter 2 we analyze the Kähler–Yang–Mills equations when a certain $SU(2)$-symmetry is imposed. The equations will reduce to new systems of equations depending on the the dimensionality of the base manifold and the rank of the vector bundles involved.

Chapter 2

Dimensional reduction of the Kähler–Yang–Mills equations

In this chapter we study dimensional reductions of the Kähler–Yang–Mills equations when imposing a particular $SU(2)$-equivariance. We do this by taking a base space of the form $X \times \mathbb{P}^1$ and a $SU(2)$-equivariant vector bundle over it. For the rest of the document we will be analyzing the equations in the context of holomorphic vector bundles, with the unknowns being the Hermitian metric and the Kähler metric, with a fixed complex structure on the base manifold. This is equivalent to the setting in Chapter 1, where we fixed the symplectic structure and the Hermitian metric and let the connection A and the complex structure J be the unknowns: a gauge transformation links the two points of view (see [15]).

2.1 Dimensional reduction and holomorphic triples

Consider a vector bundle E over a base manifold M and a compact Lie group G. We say that E is a G-**equivariant** vector bundle if there is an action of G on M that lifts to a fiberwise linear action on E. In the C^∞ setting, both actions are required to be differentiable, while if E is a holomorphic vector bundle over a complex manifold the action of G is required to be by biholomorphisms. In this case one usually considers the action of the complexification $G^{\mathbb{C}}$ of G. A G-equivariant vector bundle yields a commutative diagram of the form

$$\begin{array}{ccc} E \times G & \longrightarrow & E \\ \downarrow{\scriptstyle \pi \times \mathrm{Id}_G} & & \downarrow{\scriptstyle \pi} \\ M \times G & \longrightarrow & M. \end{array}$$

Let us consider a compact complex manifold X, although some of what follows holds when X is simply C^∞. Let us denote $M = X \times \mathbb{P}^1$. Consider the natural projections onto the first and second factor $p : M \to X$ and $q : M \to \mathbb{P}^1$. There is a natural action of the Lie group $SU(2)$ on M, acting trivially on X and in the natural way on $\mathbb{P}^1 \simeq SU(2)/U(1)$. We consider now the differentiable $SU(2)$-equivariant vector bundles over $X \times \mathbb{P}^1$. For the differentiable category, $SU(2)$-equivariant bundles are catalogued by the following result:

Proposition 2.1 (Prop. 3.1 in [16])**.** *Every differentiable $SU(2)$-equivariant vector bundle over $X \times \mathbb{P}^1$ can be equivariantly decomposed as*

$$E = \bigoplus_i p^* E_i \otimes q^* L^{\otimes n_i},$$

where E_i is a differentiable vector bundle over X, L is the line bundle over $\mathbb{P}^1$ with Chern class 1 and all $n_i \in \mathbb{Z}$ are different.

We will focus our attention to a particular subset of these equivariant vector bundles over $X \times \mathbb{P}^1$, those with an underlying C^∞ structure given by

$$E = p^*E_1 \oplus (p^*E_2 \otimes q^*L^{\otimes 2}). \tag{2.1}$$

Considering now the case when X is a compact complex manifold, we define the closely related notion of holomorphic triple over X.

Definition 2.2. A **holomorphic triple** (E_1, E_2, ϕ) over X consists of two holomorphic vector bundles E_1 and E_2 together with a sheaf homomorphism between them $\phi : E_2 \to E_1$.

For the holomorphic case, there is a one-to-one correspondence between $SU(2)$-equivariant holomorphic vector bundles over $M = X \times \mathbb{P}^1$ with underlying C^∞ structure as in (2.1) with holomorphic triples over X. This is because after fixing a $SU(2)$-invariant section η in $\Omega^{0,1}(\mathcal{O}_{\mathbb{P}^1}(-2))$, holomorphic extensions over $X \times \mathbb{P}^1$ of the form

$$0 \to p^*E_1 \to E \to p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0. \tag{2.2}$$

are determined by a second fundamental form of the type $\beta = p^*\phi \otimes q^*\eta$. Here $\mathcal{O}_{\mathbb{P}^1}(k)$ is the holomorphic line bundle over $\mathbb{P}^1$ with Chern class k. The correspondence is easily seen as follows. Extensions as above are parametrized by the sheaf cohomology group

$$\begin{aligned} H^1(X \times \mathbb{P}^1, p^*E_1 \otimes p^*E_2^* \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)) &\simeq H^0(X, E_1 \otimes E_2^*) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \\ &\simeq H^0(X, \mathrm{Hom}(E_2, E_1)) \end{aligned}$$

where we have made use of the Künneth formula and Serre duality, by which $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0))^* \simeq \mathbb{C}$. We conclude that holomorphic triples $E_2 \xrightarrow{\phi} E_1$ are in bijection with extensions of the form (2.2). These holomorphic triples will allow us to analyze the dimensional reduction of the Kähler–Yang–Mills equations in the subsequent sections.

2.2 Gravitational vortex equations

The aim of this section is to analyze the dimensional reduction of the Kähler–Yang–Mills equations (1.1) when the base space is of the form $M = X \times \mathbb{P}^1$, where X is a Kähler manifold, and we require the solution of the Kähler–Yang–Mills equations (H, Ω) to be $SU(2)$-invariant. This implies that the Kähler form is given by

$$\Omega_\sigma = p^*\omega + \sigma q^*\omega_{\mathbb{P}^1},$$

where p, q are the projections of $X \times \mathbb{P}^1$ onto the first and second factor respectively, and $\omega_{\mathbb{P}^1}$ is the Kähler form induced by the Fubini–Study metric normalized to satisfy $\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1$. Here $\sigma > 0$ is a real positive parameter.

Let us assume that we have a $SU(2)$-invariant solution (H, Ω_σ) of the Kähler–Yang–Mills equations. Since $SU(2)$ acts with different weights on each term of $p^*E_1 \oplus (p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2))$, the Hermitian metric H must decompose as

$$H = H_1 \oplus H_2 = (p^*h_1) \oplus (p^*h_2 \otimes q^*h_2'),$$

where h_1 and h_2 are Hermitian metrics on E_1 and E_2 respectively and h_2' is a $SU(2)$-invariant Hermitian metric in $\mathcal{O}_{\mathbb{P}^1}(2)$, which can be assumed to be

$$h_2' = C\frac{\mathrm{d}z \otimes \mathrm{d}\overline{z}}{(1 + |z|^2)^2}$$

in homogenous coordinates, for any arbitrary nonzero factor C. In this context, we can relate the Chern connection and curvature determined by (E, H) to the Chern connections and curvatures of each Hermitian subbundle $(p^*E_1, H_1), (p^*E_2 \otimes \mathcal{O}_{\mathbb{P}^1}(2), H_2)$. We refer to Section 1.6. of Kobayashi's book [22] for a detailed treatment of this computations. We obtain that the Chern connection induced by the Hermitian metric H is then given by

$$D_H = \begin{pmatrix} D_1 & \beta \\ -\beta^* & D_2 \end{pmatrix}. \tag{2.3}$$

Here D_1 is the Chern connection of $H_1 = p^*h_1$, D_2 is the Chern connection of $H_2 = p^*h_2 \otimes q^*h'_2$, β is the so-called second fundamental form and β^* is the corresponding adjoint map with respect to the Hermitian metrics H_1, H_2. After fixing a $SU(2)$-invariant element $\eta \in \Omega^{0,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ (which is unique up to scale factor), the second fundamental form is given by $\beta = p^*\phi \otimes q^*\eta$, where ϕ is a holomorphic section of $\mathrm{Hom}(E_2, E_1)$. For the choice of η, any multiple of $\frac{\mathrm{d}z}{(1+|z|^2)^2} \otimes \mathrm{d}\overline{z}$ is a viable candidate. We begin by calculating the trace term in the second of the KYM equations (1.1). The curvature reads as

$$F_H = \begin{pmatrix} F_{H_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{H_2} - \beta^* \wedge \beta \end{pmatrix}, \tag{2.4}$$

where D' and D'' are the induced covariant derivatives in the respective Hom bundles. The trace $\mathrm{tr}\, F_H \wedge F_H$ contains four basic terms

$$\mathrm{tr}\, F_H \wedge F_H = \mathrm{tr}(F_{H_1} - \beta \wedge \beta^*)^{\wedge 2} - \mathrm{tr}(D'\beta \wedge D''\beta^*) - \mathrm{tr}(D''\beta^* \wedge D'\beta) + \mathrm{tr}(F_{H_2} - \beta^* \wedge \beta)^{\wedge 2}$$

which we analyze separately.

First term: a quick computation over a trivialization shows that for End-valued forms the trace pairing is also cyclic in the sense that $\mathrm{tr}(A \wedge B) = (-1)^{ab}\,\mathrm{tr}(B \wedge A)$ where a and b are the degrees of A and B. Noticing that $(\beta \wedge \beta^*)^{\wedge 2}$ vanishes as it involves a 4-form over $\mathbb{P}^1$ coming from the η factor, we have

$$\begin{aligned} \mathrm{tr}(F_{H_1} - \beta \wedge \beta^*)^2 &= \mathrm{tr}(F_{H_1} \wedge F_{H1}) - \mathrm{tr}(F_{H_1} \wedge (\beta \wedge \beta^*)) - \mathrm{tr}((\beta \wedge \beta^*) \wedge F_{H_1}) + \mathrm{tr}\,\cancel{(\beta \wedge \beta^*)^2} \\ &= \mathrm{tr}(F_{H_1} \wedge F_{H_1}) - \mathrm{tr}(F_{H_1} \wedge (\beta \wedge \beta^*)) - (-1)^4\,\mathrm{tr}(F_{H_1} \wedge (\beta \wedge \beta^*)) \\ &= \mathrm{tr}(F_{H_1} \wedge F_{H_1}) - 2\,\mathrm{tr}(F_{H_1} \wedge (\beta \wedge \beta^*)). \end{aligned}$$

We can specifically choose the scale of η to satisfy $\eta \wedge \eta^* = \frac{i\sigma}{4}\omega_{\mathbb{P}^1}$ In that case,

$$\begin{aligned} \beta \wedge \beta^* &= +\frac{i}{4}p^*(\phi \circ \phi^*) \otimes q^*\sigma\omega_{\mathbb{P}^1}, \\ \beta^* \wedge \beta &= -\frac{i}{4}p^*(\phi^* \circ \phi) \otimes q^*\sigma\omega_{\mathbb{P}^1}). \end{aligned}$$

Thus $F_{H_1} \wedge (\beta \wedge \beta^*) = \frac{i}{4}p^*F_{h_1}(\phi \circ \phi^*) \otimes q^*\sigma\omega_{\mathbb{P}^1}$ and the first term is expressed as:

$$\mathrm{tr}(F_{H_1} - \beta \wedge \beta^*)^{\wedge 2} = p^*\,\mathrm{tr}(F_{h_1} \wedge F_{h_1}) - \frac{i}{2}p^*\,\mathrm{tr}(F_{h_1} \circ \phi \circ \phi^*) \otimes q^*(\sigma\omega_{\mathbb{P}^1}).$$

Second and third terms: denoting by $\partial_{1,2}$ and $\partial_{2,1}$ the corresponding Chern connections induced by h_1, h_2 in the vector bundles $\mathrm{Hom}(E_1, E_2)$ and $\mathrm{Hom}(E_2, E_1)$ over X we obtain

$$\begin{aligned} -\mathrm{tr}(D'\beta \wedge D''\beta^*) - \mathrm{tr}(D''\beta^* \wedge D'\beta) &= -\,p^*\,\mathrm{tr}(\partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*) \otimes q^*(\eta \wedge \eta^*) \\ &\quad -\,p^*\,\mathrm{tr}(\overline{\partial}_{1,2}\phi^* \wedge \partial_{2,1}\phi) \otimes q^*(\eta^* \wedge \eta) \\ &= -\,2p^*\,\mathrm{tr}(\partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*) \otimes \left(\frac{i\sigma}{4}\omega_{\mathbb{P}^1}\right) \\ &= -\,\frac{i}{2}p^*\,\mathrm{tr}(\partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*) \otimes q^*\sigma\omega_{\mathbb{P}^1}, \end{aligned}$$

by using that $\eta\wedge\eta^* = \frac{i\sigma}{4}\omega_{\mathbb{P}^1}$. The first equality follows from the fact that $D'\beta = D'(p^*\phi\otimes q^*\eta) = p^*(\partial_{2,1}\phi)\otimes\eta + p^*\phi\otimes\cancel{q^*(\partial^{1,0}_{h_2'}\eta)}$ and $D''\beta^* = p^*(\overline{\partial}_{1,2}\phi^*)\otimes q^*\eta^* + p^*\phi^*\otimes\cancel{q^*(\partial^{0,1}_{h_2'}\eta^*)}$, due to $\partial^{1,0}_{h_2'}\eta = \partial^{0,1}_{h_2'}\eta^* = 0$ which is easy to show in explicit homogenous coordinates.

Forth term: an easy computation from the explicit formula for h_2' yields the decomposition $F_{H_2} = p^*F_{h_2} - 4\pi i q^*\omega_{\mathbb{P}^1}$ which in turn implies

$$F_{H_2}\wedge F_{H_2} = p^*(F_{h_2}\wedge F_{h_2}) - 4\pi i p^*F_{h_2}\wedge q^*\omega_{\mathbb{P}^1} - 4\pi i q^*\omega_{\mathbb{P}^1}\wedge p^*F_{h_2} - 16\pi^2\cancel{q^*\omega_{\mathbb{P}^1}\wedge q^*\omega_{\mathbb{P}^1}},$$

and using $\operatorname{tr}(q^*\omega_{\mathbb{P}^1}\wedge p^*F_{h_2}) = (-1)^4\operatorname{tr}(p^*F_{h_2}\wedge q^*\omega_{\mathbb{P}^1}) = p^*\operatorname{tr}(F_{h_2})\wedge q^*\omega_{\mathbb{P}^1}$ we get:

$$\begin{aligned}\operatorname{tr}(F_{H_2}-\beta^*\wedge\beta)^2 &= \operatorname{tr}(F_{H_2}\wedge F_{H_2}) - 2\operatorname{tr}(F_{H_2}\wedge(\beta^*\wedge\beta)) + \operatorname{tr}(\cancel{(\beta^*\wedge\beta)^{\wedge 2}})\\ &= p^*\operatorname{tr}(F_{h_2}\wedge F_{h_2}) - 8\pi i p^*\operatorname{tr}(F_{h_2})\wedge q^*\omega_{\mathbb{P}^1} - 2\operatorname{tr}\left[(p^*F_{h_2} - 4iq^*\omega_{\mathbb{P}^1})\wedge(\beta^*\wedge\beta)\right]\\ &= p^*\operatorname{tr}(F_{h_2}\wedge F_{h_2}) - 8\pi i p^*\operatorname{tr}(F_{h_2})\wedge q^*\omega_{\mathbb{P}^1} - 2\operatorname{tr}(p^*F_{h_2}\wedge(\beta^*\wedge\beta)).\end{aligned}$$

Now $p^*F_{h_2}\wedge(\beta^*\wedge\beta) = -\frac{i}{4}p^*(F_{h_2}\circ(\phi^*\circ\phi))\otimes q^*\sigma\omega_{\mathbb{P}^1}$ and the fourth term is given by

$$\operatorname{tr}(F_{H_2}-\beta^*\wedge\beta)^{\wedge 2} = p^*\operatorname{tr}(F_{h_2}\wedge F_{h_2}) - \frac{8\pi i}{\sigma}p^*\operatorname{tr}F_{h_2}\otimes q^*\sigma\omega_{\mathbb{P}^1} + \frac{i}{2}p^*(F_{h_2}\circ\phi^*\circ\phi)\otimes q^*\sigma\omega_{\mathbb{P}^1}.$$

Collecting all terms yields the total trace

$$\begin{aligned}\operatorname{tr}F_H\wedge F_H = p^*(\operatorname{tr}F_{h_1}\wedge F_{h_1} + \operatorname{tr}F_{h_2}\wedge F_{h_2}) - \frac{i}{2}p^*(\operatorname{tr}F_{h_1}\circ(\phi\circ\phi^*))\otimes q^*\sigma\omega_{\mathbb{P}^1}\\ + \frac{i}{2}p^*\operatorname{tr}(F_{h_2}\circ(\phi^*\circ\phi))\otimes q^*\sigma\omega_{\mathbb{P}^1} - \frac{i}{2}p^*\operatorname{tr}(\partial_{2,1}\phi\wedge\overline{\partial}_{1,2}\phi^*)\otimes q^*\sigma\omega_{\mathbb{P}^1}\\ - \frac{8\pi i}{\sigma}p^*\operatorname{tr}F_{h_2}\otimes q^*\sigma\omega_{\mathbb{P}^1},\end{aligned}\tag{2.5}$$

and now we set out to contract this expression with the adjoint Lefschetz operator twice. As can be seen in [19], section 1.2, a decomposition of the form $\omega = \omega_1 + \omega_2$ in a direct sum of vector spaces $V_1\oplus V_2$ yields a decomposition of the form

$$\Lambda_\omega = \Lambda_1\otimes\mathrm{Id} + \mathrm{Id}\otimes\Lambda_2,$$

where Λ_ω acts on $\Lambda^k(V_1\otimes V_2)^*\simeq\bigoplus_{l+m=k}\Lambda^l V_1^*\otimes\Lambda^m V_2^*$. In particular, we get for $\Omega_\sigma = p^*\omega + q^*\sigma\omega_{\mathbb{P}^1}$

$$\Lambda_{\Omega_\sigma} = \Lambda_{p^*\omega}\otimes\mathrm{Id} + \mathrm{Id}\otimes\Lambda_{q^*\sigma\omega_{\mathbb{P}1}} = p^*\Lambda_\omega\otimes\mathrm{Id} + \mathrm{Id}\otimes q^*\Lambda_{\sigma\omega_{\mathbb{P}1}}.$$

Applying this operator in equation (2.5) and taking into account $\Lambda_{q^*\sigma\omega_{\mathbb{P}1}}(q^*\sigma\omega_{\mathbb{P}^1}) = 1$ we get:

$$\begin{aligned}\Lambda^2_{\Omega_\sigma}\operatorname{tr}F_H\wedge F_H = \left[\Lambda^2_\omega\operatorname{tr}F_{h_1}\wedge F_{h_1} + \Lambda^2_\omega\operatorname{tr}F_{h_2}\wedge F_{h_2}\right] - i\Lambda_\omega\operatorname{tr}(F_{h_1}\circ(\phi\circ\phi^*))\\ + i\Lambda_\omega\operatorname{tr}(F_{h_2}\circ(\phi^*\circ\phi)) - \frac{16\pi i}{\sigma}\Lambda_\omega\operatorname{tr}F_{h_2} - i\Lambda_\omega\operatorname{tr}(\partial_{2,1}\phi\wedge\overline{\partial}_{1,2}\phi^*)\end{aligned}\tag{2.6}$$

Here we omit the pullbacks with the understanding that functions previously defined over each factor X or $\mathbb{P}^1$ now are defined over the entire $X\times\mathbb{P}^1$. With the trace term computed we can now write the second of KYM equations. We note that the scalar curvature decomposes as $S_{\Omega_\sigma} = S_\omega + S_{\sigma\omega_{\mathbb{P}1}}$, and the second term is simply a constant. Taking all constants to the left-hand side we obtain that the second of the KYM equations (1.1) is equivalent to:

$$\begin{aligned}C = S_\omega - \alpha\Lambda^2_\omega(\operatorname{tr}F_{h_1}\wedge F_{h_1} + \alpha\operatorname{tr}F_{h_2}\wedge F_{h_2}) - \alpha\frac{16\pi i}{\sigma}\Lambda_\omega\operatorname{tr}F_{h_2}\\ - \alpha i\Lambda_\omega(\operatorname{tr}(F_{h_1}\circ\phi\circ\phi^*) - \operatorname{tr}(F_{h_2}\circ\phi^*\circ\phi) + \partial_{2,1}\phi\wedge\overline{\partial}_{1,2}\phi^*)\end{aligned}\tag{2.7}$$

We start now expanding the first equation in (1.1). The block decomposition of the curvature (2.4) leads to the system

$$\begin{aligned} i\Lambda_{\Omega_\sigma} F_{H_1} - i\beta \wedge \beta^* &= \lambda \mathrm{Id}_{p^*E_1}, \\ i\Lambda_{\Omega_\sigma} F_{H_2} - i\beta^* \wedge \beta &= \lambda \mathrm{Id}_{p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2)}, \\ \Lambda_{\Omega_\sigma} D'\beta &= 0, \\ \Lambda_{\Omega_\sigma} D''\beta^* &= 0. \end{aligned}$$

The first two equations can be written in terms of ϕ and the curvatures in the separate vector bundles E_1 and E_2:

$$\begin{aligned} i\Lambda_\omega F_{h_1} + \frac{1}{4}\phi \circ \phi^* &= \lambda \mathrm{Id}_{E_1}, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}\phi^* \circ \phi &= \left(\lambda - \frac{4\pi}{\sigma}\right) \mathrm{Id}_{E_2}. \end{aligned} \tag{2.8}$$

The last two equations are immediately satisfied as $D'\beta = p^*(\partial_{2,1}\phi) \otimes q^*\eta$ and $D''\beta = p^*(\overline{\partial}_{1,2}\phi^*) \otimes q^*\eta^*$ and thus the contraction with $\Lambda_{\Omega_\sigma} = p^*\Lambda_\omega + q^*\Lambda_{\sigma\omega_{\mathbb{P}^1}}$ vanishes due to the splitting of the 2-forms over the factors X and $\mathbb{P}^1$.

By integrating the trace of the first equation in KYM (1.1) it is possible to show that

$$\lambda = \frac{2\pi}{\mathrm{Vol}_{\Omega_\sigma}(X \times \mathbb{P}^1)} \frac{\deg_\sigma E}{\operatorname{rank} E} = \frac{2\pi}{r_1 + r_2} \left(\frac{d_1 + d_2}{\mathrm{Vol}_\omega X} + \frac{2r_2}{\sigma} \right), \tag{2.9}$$

by relating the degree of the extension E with respect to Ω_σ to the degrees d_1, d_2 of E_1, E_2 with respect to ω. This allows for rewriting the derived equations in terms of two constants $\tau = \lambda/2\pi, \tau' = \left(\lambda - \frac{4\pi}{\sigma}\right)/2\pi$ which are related to σ via

$$\sigma = \frac{2r_2 \mathrm{Vol}_\omega X}{\mathrm{Vol}_\omega X (r_1 + r_2)\tau - (d_1 + d_2)}, \tag{2.10}$$

and satisfying the relation

$$\mathrm{Vol}_\omega X (r_1\tau + r_2\tau') = (d_1 + d_2), \tag{2.11}$$

implying that h_1 and h_2 satisfy

$$\begin{aligned} i\Lambda_\omega F_{h_1} + \frac{1}{4}\phi \circ \phi^* &= 2\pi\tau \mathrm{Id}_{E_1}, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}\phi^* \circ \phi &= 2\pi\tau' \mathrm{Id}_{E_2}, \end{aligned} \tag{2.12}$$

which are known as the *coupled vortex equations.* They were introduced and extensively studied by García-Prada in [16]. Together with the equation coupling to the Kähler metric (2.7) we state them together and we will be referring to them as the *gravitational vortex equations*:

$$\begin{cases} i\Lambda_\omega F_{h_1} + \frac{1}{4}\phi \circ \phi^* = 2\pi\tau \mathrm{Id}_{E_1}, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}\phi^* \circ \phi = 2\pi\tau' \mathrm{Id}_{E_2}, \\ S_\omega - \alpha\Lambda_\omega^2 \{\operatorname{tr} F_{h_1} \wedge F_{h_1} + \operatorname{tr} F_{h_2} \wedge F_{h_2}\} - \alpha\frac{16\pi i}{\sigma}\Lambda_\omega \operatorname{tr} F_{h_2} \\ \qquad -\alpha i \Lambda_\omega \left\{\operatorname{tr}(F_{h_1} \circ \phi \circ \phi^*) - \operatorname{tr}(F_{h_2} \circ \phi^* \circ \phi) + \partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*\right\} = C. \end{cases} \tag{2.13}$$

We have just proved the following.

Proposition 2.3. *Let $T = (E_1, E_2, \phi)$ be a holomorphic triple over a compact Kähler manifold (X, ω) and let E be the holomorphic vector bundle over $X \times \mathbb{P}^1$ defined by* (2.2)*. Let σ, τ, τ' be related by* (2.10) *and* (2.11)*. The following are equivalent:*

- *E admits a $SU(2)$-invariant solution to the Kähler–Yang–Mills equations* (1.1).
- *The bundles E_1 and E_2 admit solutions to the gravitational vortex equations* (2.13).

As noted above, these equations contain the coupled vortex equations, for which there are already results characterizing the existence of solutions in particular situations. The next sections will introduce additional hypothesis on the dimensionality of the base manifold and the ranks of the vector bundles involved in order to further simplify the equations.

2.3 Equations on line bundles

We analyze the particular situation in which the vector bundles E_1 and E_2 in the holomorphic triple are line bundles, temporarily denoted as L_1 and L_2 ($\operatorname{rank} L_1 = \operatorname{rank} L_2 = 1$). We still assume that X is a Kähler manifold of arbitrary complex dimension.

The first thing to note is that $\operatorname{End}(L_1)$ and $\operatorname{End}(L_2)$ are trivial bundles. This is because for line bundles L over X, tensoring with the dual gives $L \otimes L^* = X \times \mathbb{C}$, as can be seen by multiplying the transition functions (which are one-by-one complex matrices):

$$g_{11,L} \cdot g_{11,L^*} = g_{11,L} \cdot (g^t)^{-1}_{11,L} = 1.$$

The triviality of $\operatorname{End} L_i$ implies that the curvatures F_{h_1} and F_{h_2} are simply 2-forms over X. In this setting we assume that L_2 is the trivial line bundle by tensoring the extension

$$0 \to p^* L_1 \to E \to p^* L_2 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

with $p^* L_2^*$. This implies that $\phi \circ \phi^* = \phi^* \circ \phi = |\phi|^2$, the squared norm of $\phi \in H^0(X, L)$ with respect to the Hermitian metric $h = h_1 \otimes h_2^*$ on $L = L_1 \otimes L_2^*$. The *gravitational vortex equations* can be thus written for line bundles as

$$\begin{cases} i\Lambda_\omega F_{h_1} + \frac{1}{4}|\phi|^2 = 2\pi\tau \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}|\phi|^2 = 2\pi\tau', \\ S_\omega - \alpha\Lambda_\omega^2 (F_{h_1} \wedge F_{h_1} + F_{h_2} \wedge F_{h_2}) - \frac{16\pi i\alpha}{\sigma}\Lambda_\omega F_{h_2} \\ \qquad -\alpha i \Lambda_\omega \left(F_{h_1}|\phi|^2 - F_{h_2}|\phi|^2 + \partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^* \right) = C. \end{cases} \tag{2.14}$$

These expressions can be reduced by means of a Weitzenböck-type formula

$$\Delta|\phi|^2 = 2i\Lambda_\omega \left(F_h|\phi|^2 + \partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^* \right),$$

and by direct substitution of $\Lambda_\omega F_{h_2}$ from the second into the third equation to arrive at

$$\begin{cases} i\Lambda_\omega F_{h_1} + \frac{1}{4}|\phi|^2 = 2\pi\tau, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}|\phi|^2 = 2\pi\tau', \\ S_\omega - \alpha\Lambda_\omega^2 (F_{h_1} \wedge F_{h_1} + F_{h_2} \wedge F_{h_2}) \\ \qquad -\frac{\alpha}{2}(\Delta + \tilde{\tau})(|\phi|^2 - \tilde{\tau}) = C'. \end{cases}$$

by defining $\tilde{\tau} = \frac{8\pi}{\sigma}$. Here $C' = C - 2\tilde{\tau}\alpha\lambda$, another constant. Substracting the second equation from the first we can get an expression involving the Hermitian metric h. However, the equation coupling the Hermitian and Kähler metrics cannot be fully stated with respect to h alone. This is due to the presence of the quadratic terms. Certainly

$$F_h \wedge F_h = F_{h_1} \wedge F_{h_1} - F_{h_1} \wedge F_{h_2} - F_{h_2} \wedge F_{h_1} + F_{h_2} \wedge F_{h_2} = F_{h_1} \wedge F_{h_1} + F_{h_2} \wedge F_{h_2} - 2F_{h_1} \wedge F_{h_2,}$$

since $F_{h_2} \wedge F_{h_1} = (-1)^{2 \cdot 2} F_{h_1} \wedge F_{h_2}$, and therefore the cross term $F_{h_1} \wedge F_{h_2}$ prevents this to be written solely in terms of h.

2.4 Abelian gravitational vortex equations on Riemann surfaces

Now it is possible to analyze the situation when X is a Riemann surface ($\dim_{\mathbb{C}} X = 1$) and the vector bundles involved are still line bundles L_1, L_2 over X. The equations will simplify greatly. We consider again the line bundle $L = L_1 \otimes L_2^*$.

As in the previous Section 2.3, the endomorphism bundles EndL_1 and EndL_2 are trivial, and we have that $\phi \circ \phi^*$ is the endomorphism of the line bundle given by scalar multiplication by $|\phi|^2$, the squared norm of ϕ with respect to h, the Hermitian metric in L given by $h = h_1 \otimes h_2^*$. In this particular case the quadratic terms $F_{h_1} \wedge F_{h_1}, F_{h_2} \wedge F_{h_2}$ vanish identically since there are no nontrivial 4-forms over a Riemann surface. This allows to write the previously obtained equation in terms of h alone. The Chern connection determined by (L, h) has a curvature 2-form given by $F_h = F_{h_1} - F_{h_2}$. The trace term $\operatorname{tr} F_H \wedge F_H$ from (2.7) therefore reduces, by defining $\tilde{\tau} = \frac{8\pi}{\sigma}$, to

$$\Lambda^2_{\Omega_\sigma} \operatorname{tr} F_H \wedge F_H = -2i\tilde{\tau} p^* \Lambda_\omega F_{h_2} - ip^* \Lambda_\omega (F_h |\phi|^2 + \partial\phi \wedge \overline{\partial}\phi^*) = -2i\tilde{\tau}\Lambda_\omega F_{h_2} - \frac{1}{2}\Delta|\phi|^2,$$

where we are omitting pullbacks and making use of the Weitzenböck-type equality

$$\Delta|\phi|^2 = 2i\Lambda_\omega \overline{\partial}\partial|\phi|^2. = 2i\Lambda_\omega(|\phi|^2 F_h + \partial\phi \wedge \overline{\partial}\phi^*).$$

From equations (2.8) and $\tilde{\tau} = \frac{8\pi}{\sigma}$ we obtain

$$\begin{aligned} i\Lambda_\omega F_{h_1} + \frac{1}{4}|\phi|^2 &= \lambda, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}|\phi|^2 &= \lambda - \frac{\tilde{\tau}}{2} \end{aligned} \tag{2.15}$$

By substracting these two identities and writing everything in terms of F_h we conclude that

$$i\Lambda_\omega F_h + \frac{1}{2}\left(|\phi|^2 - \tilde{\tau}\right) = 0, \tag{2.16}$$

which is precisely the $\tilde{\tau}$-vortex equation, introduced by Landau and Ginzburg [23] in the study of superconductivity and studied by García-Prada in [15]. For the equation coupling the Hermitian metric and the Kähler metric (2.7) we obtain by substituting $\Lambda_\omega F_{h_2}$ from the second of (2.15) and implementing the above remarks that

$$\begin{aligned} C &= S_\omega + \frac{\alpha}{2}\Delta|\phi|^2 + 2\alpha\tilde{\tau}\left(\lambda - \frac{\tilde{\tau}}{2} + \frac{1}{4}|\phi|^2\right), \\ C' &= S_\omega + \frac{\alpha}{2}\Delta|\phi|^2 + 2\alpha\tilde{\tau}\lambda - \alpha\tilde{\tau}^2 + \alpha\frac{\tilde{\tau}}{2}|\phi|^2, \end{aligned}$$

or regrouping constants in the right-hand side,

$$S_\omega + \frac{\alpha}{2}(\Delta + \tilde{\tau})(|\phi|^2 - \tilde{\tau}) = C. \tag{2.17}$$

Collecting (2.16) and (2.17) together yields the Abelian gravitational vortex equations:

$$\begin{cases} i\Lambda_\omega F_h + \frac{1}{2}\left(|\phi|^2 - \tilde{\tau}\right) = 0, \\ S_\omega + \frac{\alpha}{2}(\Delta + \tilde{\tau})(|\phi|^2 - \tilde{\tau}) = C \end{cases} \tag{2.18}$$

These equations have been studied in [2] and for the particular case of $C = 0$ they are related to the physics of cosmic strings. The authors refer to these as the *gravitating vortex equations*. Ours differ by a factor 1/2 appearing in (2.16), and it is due to the particular choice of the normalization of the element $\eta \in \Omega^{0,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$, being in any case equivalent to the equations presented in [2].

We have just showed that a $SU(2)$-equivariant solution (H, Ω_σ) of the KYM equations on $X \times \mathbb{P}^1$ determines a solution to the Abelian gravitational vortex equations for $\tilde{\tau} = 8\pi/\sigma$. The process can be reversed in the following manner. Suppose that (h, ω) is a solution to the Abelian gravitational vortex equations, for a Riemann surface X and a line bundle L over it. Then take a rank-two vector bundle extending p^*L by $p^*\mathcal{O}_X \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2)$, the isomorphism class of the holomorphic extension determined by $\phi \in H^0(X, L)$. Then we propose a Kähler form given by $\Omega_\sigma = p^*\omega + q^*\sigma\omega_{\mathbb{P}^1}$ over the Kähler manifold $X \times \mathbb{P}^1$ and a Hermitian metric

$$H = H_1 \oplus H_2 = p^*h_1 \oplus (p^*h_2 \otimes q^*h_2')$$

for Hermitian metric h_1, h_2 on L and $\mathcal{O}_X$ respectively, and h_2' the $SU(2)$-invariant metric on $\mathcal{O}_{\mathbb{P}^1}(2)$ determined in homogenous coordinates by $h_2' = \frac{dz \otimes d\bar{z}}{(1+|z|^2)^2}$. The Kähler–Yang–Mills equations are translated by the above dimensional reduction process into

$$\begin{aligned} i\Lambda_\omega F_{h_1} + \frac{1}{4}|\phi|^2 &= \lambda, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}|\phi|^2 &= \lambda - \frac{\tilde{\tau}}{2}, \\ \Lambda_{\Omega_\sigma} D'\beta &= 0, \\ \Lambda_{\Omega_\sigma} D''\beta &= 0, \\ S_\omega + \alpha(\Delta_\omega|\phi|^2 + 2i\tau\Lambda_\omega F_{h_2}) &= c. \end{aligned}$$

The third and fourth equations are trivially satisfied as in Section 2.2. The second and fifth are solved by a Hermitian metric $h_2 = e^f$ on $\mathcal{O}_X$ for a function f solving the Poisson equation $\Delta f = \frac{1}{2}|\phi|^2 - \tilde{\tau} + 2\lambda$. By Hodge theory, this equation admits a solution if and only if the righ-hand side integrates to zero over X. But integrating $\frac{1}{2}|\phi|^2$ in terms of ΛF_h shows that the vanishing of the right-hand-side is equivalent to (2.9). A quick computation shows that the first equation is solved by the metric $h_1 = h \otimes h_2$, and this finishes the proof of the correspondence.

The existence of solutions to the Abelian gravitational vortex equations on Riemann surfaces has been studied in [2],[4],[14]. In particular, [2] gives an existence theorem for genus $g \geq 1$. In this case, a deformation method yields a solution for small values of the coupling parameter α.

In order to describe what happens in genus $g = 0$, we note that via integration of the equations (2.18) it can be shown that the constant c appearing there is topological in nature, being given by

$$c = \frac{2\pi(\chi(X) - 2\alpha\tilde{\tau}c_1(L))}{\operatorname{Vol}_\omega X}$$

and thus completely determined by the first Chern class of the line bundle, the genus of the Riemann surface X and the cohomology class of the Kähler form ω. When the topological constant c vanishes and $c_1(L) > 0$ the only possible topology for X is that of the Riemann sphere $g = 0$, and the Abelian gravitational vortex equations are equivalent to the Einstein–Bogomol'nyi equations. These are related to the physics of cosmic strings, and the solutions to these are called Nielsen–Olesen strings [28].

Yang proved a sufficiente condition [33, 34] for the existence of Nielsen–Olesen strings on $\mathbb{P}^1$ in terms of the relative position of the zeros of the Higgs field ϕ. In [2], this condition is translated into algebraic terms, and subsequent articles [4, 14] prove that the existence of solutions to the Einstein-Bogomol'nyi equations is completely characterized by a certain notion of GIT stability of the divisor corresponding to (L, ϕ). We recall that the moduli space of effective divisors parametrizes these pairs (L, ϕ), and the group $SL(2, \mathbb{C})$, which are automorphisms of $\mathbb{P}^1$, acts naturally on this moduli space giving rise to a Geometric Invariant Theory quotient. In [14], García-Fernández, Pingali and Yau prove that for positive values of the constant c, the existence of solutions to the general Abelian gravitational vortex equation with prescribed volume in $\mathbb{P}^1$ is related again to GIT polystability.

2.5 Equations on Riemann surfaces

We analyze now the gravitational vortex equations when the base space is a compact Riemann surface $\dim X_{\mathbb{C}} = 1$ but the vector bundles E_1 and E_2 are of arbitrary rank r_1 and r_2 respectively. We will obtain equations generalizing those of the previous section. The dimensionality of X again implies that the quadratic terms $F_{h_1} \wedge F_{h_1}$ and $F_{h_2} \wedge F_{h_2}$ vanish identically. From the general gravitational vortex equations (2.13) we obtain

$$\begin{cases} i\Lambda_\omega F_{h_1} + \frac{1}{4}\phi \circ \phi^* = 2\pi\tau \mathrm{Id}_{E_1}, \\ i\Lambda_\omega F_{h_2} - \frac{1}{4}\phi^* \circ \phi = 2\pi\tau' \mathrm{Id}_{E_2}, \\ S_\omega - \frac{16\pi i\alpha}{\sigma}\Lambda_\omega \operatorname{tr} F_{h_2} \\ \qquad -\alpha i\Lambda_\omega \left[\operatorname{tr}(F_{h_1} \circ \phi \circ \phi^*) - \operatorname{tr}(F_{h_2} \circ \phi^* \circ \phi) + \operatorname{tr}(\partial_{2,1}\phi \wedge \overline{\partial}_{1,2}\phi^*)\right] = C. \end{cases} \tag{2.19}$$

The first two equations are simply the coupled vortex equations. The higher rank of the vector bundles involved makes the third equation in (2.19) not so easy to simplify in terms of the Laplacian operator.

The existence of solutions to the coupled vortex equations over compact Riemann surfaces has been studied in [16] when E_2 is assumed to be a line bundle, and later generalized for arbitrary ranks in [7]. These previous results link the existence of solutions to the coupled vortex equations to a certain notion of stability of the triple (E_1, E_2, ϕ). This turns out to be equivalent to the stability of the extension given by (2.2). We will introduce the appropriate notions of Geometric Invariant Theory and stability of holomorphic triples in the following chapter. The aim is to combine what is known for the coupled vortex equations and the existence results given for the Abelian case to formulate a conjecture regarding the existence of solutions to (2.19) for the particular case of genus $g = 0$, one of the main objectives of the present work. These are nothing but the very first steps towards a deeper understanding of the gravitational vortex equations on general Riemann surfaces.

Chapter 3

Holomorphic triples and stability

3.1 Stability and Hitchin–Kobayashi correspondences

In this section we recall some previous results concerning existence of solutions to the Hermitian–Yang–Mills equations in terms of the stability of the vector bundles involved. This is of paramount importance as the Hermitian–Yang–Mills equations (and its dimensional reductions) are a subset of the Kähler–Yang–Mills equations (and its dimensional reductions respectively). Throughout this section we let (X, ω) denote a compact Kähler manifold of arbitrary complex dimension n. We recall that the degree of any vector bundle E over X with respect to the Kähler form ω is an integer number obtained by integrating the first Chern class against the $(n-1)^{\text{th}}$ power of the Kähler form:

$$\deg_\omega E = \frac{1}{(n-1)!}\int_X c_1(E) \wedge \omega^{n-1} = \frac{i}{2\pi}\int_X \Lambda_\omega \operatorname{tr} F_A \frac{\omega^n}{n!}, \tag{3.1}$$

where A is any connection on E. Note that for Riemann surfaces the degree is independent of the Kähler structure, and in that case we write simply $\deg E$. With this concept of degree we can define the standard notion of stability of a vector bundle.

Definition 3.1. Let (X, ω) be a compact Kähler manifold surface and E a holomorphic bundle over M. E is said to be stable if for every non-trivial coherent subsheaf $E' \subset E$

$$\mu(E') < \mu(E)$$

where $\mu(E') := \frac{\deg_\omega E'}{\operatorname{rank} E'}$ is called the slope of the bundle E' with respect to ω. If one replaces the strict inequality by $\mu(E') \leq \mu(E)$ then E is said to be semistable. A vector bundle E is said to be polystable if it is isomorphic to a direct sum of holomorphic stable bundles all with the same slope as E.

A nice feature of Riemann surfaces is that each subsheaf E' of a vector bundle E, which is a locally free coherent sheaf, is necessarily torsion-free, and there is a unique vector subbundle $E'' \subset E$ with $\operatorname{rank} E'' = \operatorname{rank} E'$ fitting the short exact sequence

$$0 \to E' \to E'' \to E''/E' \to 0,$$

where E''/E' is a pure torsion sheaf. This implies in particular that $\deg E' = \deg E'' - \deg E''/E' \leq \deg E''$, and thus stability of bundles over Riemann surfaces need only be checked with respect to holomorphic subbundles $E'' \subset E$. In complex dimension 2 we can also restrict our attention to locally free coherent subsheaves E', even though they are not necessarily subbundles of E.

Stability of bundles is related to existence of solutions to the Hermitian–Yang–Mills equations via the Hitchin–Kobayashi correspondence, proved by Donaldson in the algebraic setting [9], and by Uhlenbeck and Yau for general compact Kähler manifolds [32].

Theorem 3.2 (Donaldson–Uhlenbeck–Yau)**.** *Let E be a holomorphic bundle over a compact Kähler manifold (X, ω). Then E admits a solution h to the Hermitian–Yang–Mills equations*

$$i\Lambda_\omega F_h = \lambda Id_E$$

if and only if E is polystable.

If we assume that X is a Riemann surface, Hermitian–Yang–Mills connections correspond to projectively flat structures on the vector bundle E over X. Through the holonomy representation, this allows an interpretation of the above result in terms of the Narasimhan–Seshadri theorem: irreducible projective unitary representations of the fundamental group $\pi_1(X)$ are in one-to-one correspondence with stable holomorphic bundles over X.

A similar notion of stability is defined in the case of G-equivariant vector bundles, which is relevant for the study of dimensional reductions. In this context, a vector bundle E is said to be G-invariantly stable if

$$\mu(E') < \mu(E)$$

for every G-equivariant subsheaf $E' \subset E$. Similarly one defines G-invariant semistability and polystability. This notion of G-invariant stability has proven useful to characterize the existence of solutions to several equations obtained by dimensional reduction from the Hermitian–Yang–Mills equations, obtaining new versions of the Hitchin–Kobayashi correspondence.

Two main results (Theorems 4 and 5 in [15]) established a G-invariant Hitchin–Kobayashi correspondence as follows.

Theorem 3.3. *Let E be a G-equivariant holomorphic vector bundle over a compact Kähler manifold (X, ω). If E has a G-invariant Hermitian–Yang–Mills metric h then $(E, h) = \bigoplus_i (E_i, h_i)$ where E_i is G-invariantly stable having a G-invariant Hermitian–Yang–Mills metric h_i and $\mu(E_i) = \mu(E)$.*

Theorem 3.4. *Let E be a G-equivariant holomorphic vector bundle over a compact Riemann surface. If E is G-invariantly stable, then it supports a G-invariant Hermitian–Yang–Mills metric.*

3.2 Holomorphic triples and coupled vortices

We now consider a compact Riemann surface X. A holomorphic triple $T = (E_1, E_2, \phi)$ consists of two holomorphic vector bundles over X and a holomorphic map between them $\phi : E_2 \to E_1$. The tuple $(r_1, r_2, d_1, d_2) = (\operatorname{rank} E_1, \operatorname{rank} E_2, \deg E_1 \deg E_2)$ is oftern referred to as the *type* of the triple. We now define the notion of σ-stability of a triple.

Definition 3.5. Let $T = (E_1, E_2, \phi)$ be holomorphic triple. A subtriple $T' = (E_1', E_2', \phi')$ is a set of coherent subsheaves $E_i' \subset E_i$ and a sheaf map ϕ' making the following diagram commutative.

$$\begin{array}{ccc} E_2' & \xrightarrow{\phi'} & E_1' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ E_2 & \xrightarrow{\phi} & E_1 \end{array}$$

The subtriple T' is said to trivial if $T' = (0,0,0)$ or $T' = T$. The σ-degree and σ-slope are defined by

$$\deg_\sigma(T') := \deg(E_1' \oplus E_2') + r_2'\sigma,$$
$$\mu_\sigma(T') := \frac{\deg_\sigma T'}{r_1' + r_2'}.$$

The triple is then said to be **σ-stable** if $\mu_\sigma(T') < \mu_\sigma(T)$ for every non-trivial subtriple. The analogous concept of **σ-semistability** is defined by replacing the previous strict inequality by a weak one.

Again it suffices to check the inequalities for vector subbundles, since saturated subsheaves (those with torsionless quotient sheaf) in Riemann surfaces are precisely vector subbundles. For any holomorphic triple $T = (E_1, E_2, \phi)$ there is a dual triple given by $T^* = (E_2^*, E_1^*, \phi^*)$ induced by the isomorphism

$$\mathrm{Hom}(E_2, E_1) \simeq \mathrm{Hom}(E_1^*, E_2^*),$$

and moreover, a straightforward computation shows that the σ-stability of T is equivalent to the σ-stability of its dual triple T^*. Another relevant concept is that of quotient triple. Consider a subtriple $T' = (E_1, E_2, \phi)$ of type (r_1', r_2', d_1', d_2') of $T = (E_1, E_2, \phi)$. Then we can construct the quotient triple defined as $T'' = (E_1/E_1', E_2/E_2', \phi_q)$, where the quotient bundles fit the short exact sequence

$$0 \to E_i' \to E_i \to E_i/E_i'' \to 0,$$

and therefore give rise to a commutative diagram of triples

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_2' & \longrightarrow & E_2 & \longrightarrow & E_2/E_2' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_q} & & \\
0 & \longrightarrow & E_1' & \longrightarrow & E_1 & \longrightarrow & E_1/E_1' & \longrightarrow & 0.
\end{array}$$

Proposition 3.6. *Let T' be a subtriple of T as above. If T is σ-stable then*

$$\mu_\sigma(T') < \mu_\sigma(T) < \mu_\sigma(T'')$$

where T'' is the quotient triple induced by T'. If T is σ-semistable, the analogous weak inequality holds.

Proof. By exactness of the sequence $0 \to E_i' \to E_i \to E_i/E_i'' \to 0$, we have that $d_i'' = d_i - d_i'$ and $r_i'' = r_i - r_i'$, where $d_i, d_i', d_i'', r_i, r_i', r_i''$ stand for the degrees and ranks of $E_i, E_i', E_i'' = E_i/E_i'$ respectively. Therefore,

$$\begin{aligned}
\mu_\sigma(T'') = \frac{d_1'' + d_2'' + r_2''\sigma}{r_1'' + r_2''} &= \frac{d_1 + d_2 + \sigma r_2}{r_1 + r_2 - r_1' - r_2'} - \frac{d_1' + d_2' + r_2'}{r_1 + r_2 - r_1' - r_2'} \\
&= \mu_\sigma(T)\frac{r_1 + r_2}{r_1 + r_2 - r_1' - r_2'} - \mu_\sigma(T')\frac{r_1' + r_2'}{r_1 + r_2 - r_1' - r_2'} > \mu_\sigma(T) > \mu_\sigma(T'),
\end{aligned}$$

where we have used that $\mu_\sigma(T) > \mu_\sigma(T')$. □

When τ and σ are related by (2.10) we speak of σ- and τ-stability of triples indistinctly. Polystability is defined in terms of the parameter τ.

Definition 3.7. The triple $T = (E_1, E_2, \phi)$ is said to be reducible if there are direct sum decompositions $E_1 = \bigoplus_{i=1}^k E_{1i}, E_2 = \bigoplus_{i=1}^k E_{2i}$ and $\phi = \bigoplus_{i=1}^k \phi_i$ such that $\phi_i \in \mathrm{Hom}(E_{2i}, E_{1i})$, and we write $T = \bigoplus_{i=1}^k T_i$ for $T_i = (E_{1i}, E_{2i}, \phi)$. A reducible triple is said to be τ-**polystable** if the following holds, for the value of σ given by (2.10) and τ' related to τ by (2.11):

1. Each ϕ_i is non-trivial unless $E_{1i} = 0$ or $E_{2i} = 0$.
2. if $\phi_i \neq 0$ then T_i is σ-stable.
3. if $E_{1i} = 0$ then E_{2i} is a stable bundle of slope τ',
4. if $E_{2i} = 0$ then E_{1i} is a stable bundle of slope τ.

The main result in [7] relates (poly)-stability of triples to $SU(2)$-invariant (poly)-stability.

Theorem 3.8 (Theorems 4.1 and 4.7 in [7]). *Let $T = (E_1, E_2, \phi)$ be a holomorphic triple over a compact Riemann surface, and let E be the holomorphic extension* (2.2) *over $X \times \mathbb{P}^1$ induced by T. Then:*

1. *If E_1 and E_2 are not isomorphic, then T is σ-stable if and only if E is stable with respect to Ω_σ.*
2. *If $E_1 \simeq E_2 \simeq F$, then T is σ-stable if and only if E decomposes as a direct sum*

$$E = (p^*F \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \oplus (p^*F \otimes \mathcal{O}_{\mathbb{P}^1}(1))$$

   and $(p^*F \otimes \mathcal{O}_{\mathbb{P}^1}(1))$ is stable with respect to Ω_σ.

3. *T is a τ-polystable triple if and only if E is $SU(2)$-invariantly polystable with respect to Ω_σ.*

These results together with the invariant version of the Hitchin–Kobayashi correspondence give the basic existence result of the coupled vortex equations.

Theorem 3.9 (Theorem 5.1 in [7]). *Let $T = (E_1, E_2, \phi)$ be a holomorphic triple over a compact Riemann surface. Then the following are equivalent.*

1. *The bundles support Hermitian metrics h_1, h_2 solving the coupled vortex equations* (2.12)*.*
2. *The triple T is τ-polystable.*

Due to the role of stable triples in the study of solutions to the coupled vortex equations it is natural to wonder about the moduli space of these stable triples under the following equivalence relation. Consider two triples $T = (E_1, E_2, \phi), T' = (E_1', E_2', \phi')$. T and T' are said to be isomorphic if there exist isomorphisms of bundles $u : E_1 \to E_1'$ and $v : E_2 \to E_2'$ such that $\phi' \circ v = u \circ \phi$. Let $\mathcal{M}(t)$ be the set of equivalence classes of holomorphic triples of type $t = (r_1, r_2, d_1, d_2)$, and let $\mathcal{M}_\tau(t) \subset \mathcal{M}(t)$ be the subset of equivalence classes of τ-stable triples of the same type. This moduli space has the structure of an algebraic variety as is shown in [7].

Theorem 3.10 (Theorem 6.1 in [7]). *Let X be a compact Riemann surface of genus g. The moduli space of τ-stable triples of type (r_1, r_2, d_1, d_2) is a complex analytic space with a natural Kähler structure outside of the singularities, with dimension at a smooth point given by*

$$1 + r_2 d_1 - r_1 d_2 + (r_1^2 + r_2^2 - r_1 r_2)(g - 1), \tag{3.2}$$

and it is non-empty if and only if τ lies in the interval $I = (d_1/r_1, \mu_M)$, where

$$\mu_M = \frac{d_1}{r_1} + \frac{r_2}{|r_1 - r_2|}\left(\frac{d_1}{r_1} - \frac{d_2}{r_2}\right)$$

if $r_1 \neq r_2$ and $\mu_M = \infty$ if $r_1 = r_2$. Moreover $\mathcal{M}_\tau(t)$ is a quasi-projective varity and projective if $r_1 + r_2$ and $d_1 + d_2$ are coprime and τ is generic.

The set of critical values for τ for the last condition to fail is a finite set contained within the interval I.

3.3 Geometric Invariant Theory

We make a brief pause to introduce basic concepts of Geometric Invariant Theory which will allow us to formulate a conjecture regarding the existence of solutions to the gravitational vortex equations in terms of an action on the moduli space of stable triples. We cite Mumford's book [26] as a basic source. Consider a quasi-projective variety $\mathcal{M}$ and a reductive group G acting on it. Consider a G-equivariant line bundle L over $\mathcal{M}$, i.e. a lift of the action which is fiberwise linear.

Consider the space of sections $H^0(X, L^{\otimes k})$ for any $k \geq 0$, and the subspace of G-invariant sections $H^0(\mathcal{M}, L^{\otimes k})^G$. A point $p \in \mathcal{M}$ is said to be **GIT semistable** if there exists some $k \geq 0$ and some G-invariant section $f \in H^0(\mathcal{M}, L^{\otimes k})^G$ such that:

1. $f(p) \neq 0$
2. The open subset $\mathcal{M}_f = \{q \in \mathcal{M} : f(q) \neq 0\}$ is affine.

A semistable point is said to be **GIT stable** if furthermore the action of G on $\mathcal{M}_f$ is closed and the stabilizer of G at x is finite. Dropping the last hypothesis yields the notion of **GIT polystable** points. We denote by $\mathcal{M}^s$ and $\mathcal{M}^{ss}$ the set of stable and semistable points respectively. The GIT quotient is defined as

$$\mathcal{M} /\!\!/ G := \operatorname{Proj}\left(\bigoplus_{k\geq 0} H^0(\mathcal{M}, L^{\otimes k})\right),$$

and there is a surjective G-invariant rational map $\pi : \mathcal{M}^{ss} \to \mathcal{M} /\!\!/ G$. Furthermore, when restricted to stable points, this map is a geometric quotient, meaning that the image of $\mathcal{M}^s$ under π, $\mathcal{M}^s /\!\!/ G := \pi(\mathcal{M}^s)$, coincides with the quotient under the group action

$$X^s /\!\!/ G = X^s/G.$$

3.4 Automorphisms of Riemann surfaces and moduli space of triples

We now return to holomorphic triples over Riemann surfaces. Consider the automorphism group $\mathrm{Aut}(X)$ of a Riemann surface. We define an action of $\mathrm{Aut}(X)$ on the set of holomorphic triples by pullback: if $T = (E_1, E_2, \phi)$ is a holomorphic triple then

$$g \cdot T := (g^*E_1, g^*E_2, g^*\phi),$$

where we recall that the pullback of the bundle is given by $g^*E_i = \{(x, e_i) \in X \times E_i : g \cdot x = \pi(e_i)\}$ and that the pullback map is $g^*\phi(x, e_2) := (x, \phi(e_2))$.

Proposition 3.11. *The action of the automorphism group preserves equivalence classes of triples and τ-stability.*

Proof. The first assertion follows from the functorial properties of the pullback. If T' and T are equivalent triples then there are isomorphisms u, v making the first diagram commute, and from this follows the commutativity of the second.

$$\begin{array}{ccc} E_2 & \xrightarrow{\phi} & E_1 \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ E_2' & \xrightarrow{\phi'} & E_1' \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} g^*E_2 & \xrightarrow{g^*\phi} & g^*E_1 \\ {\scriptstyle g^*u}\uparrow & & \uparrow{\scriptstyle g^*v} \\ g^*E_2' & \xrightarrow{g^*\phi'} & g^*E_1'. \end{array}$$

The degree of the pullback under g of the bundle is the degree of the original bundle times the degree (as a map) of g. Since $g \in \mathrm{Aut}(X)$, it is an orientation-preserving diffeomorphism and therefore has degree 1. Therefore $\deg(g^*E_i) = \deg(E_i)$. Since the pullback preserves rank, this implies that $\mu_\sigma(T') = \mu_\sigma(g^*T')$. The second assertion follows from this and the fact that there is a one-to-one correspondence between subtriples of T and subtriples of the pullback triple g^*T. □

The above proposition implies that the automorphism group of a Riemann surface acts in the moduli space of stable holomorphic triples in a well defined way. Let us now consider the case of the projective line $\mathbb{P}^1$. We recall that $\mathbb{P}^1 \simeq SL(2,\mathbb{C})/P$ where P is the parabolic subgroup of upper triangular matrices. Therefore the Lie group $SL(2,\mathbb{C})$ acts naturally on $\mathbb{P}^1$ by biholomorphisms, which is to say that there is an injection $SL(2,\mathbb{C}) \hookrightarrow \mathrm{Aut}(X)$. Since we know that the moduli space of stable triples over a Riemann surface is a quasi-projective variety we are in position to consider GIT stability and GIT quotients for this $SL(2,\mathbb{C})$ action.

Inspired by the previous existence results for the Abelian gravitational vortex equations we state the following conjecture regarding the general gravitational vortex equations on $\mathbb{P}^1$.

Conjecture 3.12. *Let $\mathbb{P}^1$ be the projective line and $T = (E_1, E_2, \phi)$ a holomorphic triple over $\mathbb{P}^1$ of type $t = (n_1, n_2, d_1, d_2)$. Let σ, τ, τ' be related as in* (2.10),(2.11). *The following are equivalent.*

1. *There exists a solution to the gravitational vortex equations* (2.19) *on* $(\mathbb{P}^1, T)$.

2. *The triple T is τ-polystable and its equivalence class is furthermore GIT-polystable for the action of $SL(2,\mathbb{C})$ on $\mathcal{M}_\tau(t)$.*

The motivation behind this conjecture lies at the identification of (isomorphism classes of) line bundles with effective divisors over $\mathbb{P}^1$. Previous results [2,4,14] characterize the existence of Abelian gravitational vortices on $(\mathbb{P}^1, L)$ for a line bundle in terms of stability under a GIT-action of $SL(2,\mathbb{C})$, and it seems possible that this characterization holds for higher ranks and the corresponding equations.

3.5 Stable triples over $\mathbb{P}^1$

In this section we outline some previous descriptions of the moduli spaces of stable triples over $\mathbb{P}^1$ as a first approach towards understanding the solutions to the gravitational vortex equations. One of the main advantages to working with $\mathbb{P}^1$ is the following theorem:

Theorem 3.13 (Grothendick, [17]). *Let E be a holomorphic vector bundle of rank r over $\mathbb{P}^1$. Then there exists $n_1, \ldots, n_r \in \mathbb{Z}$ such that*

$$E \simeq \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r).$$

Here $\mathcal{O}(d)$ denotes the line bundle over $\mathbb{P}^1$ of degree d.

We start by analyzing holomorphic triples of line bundles and build up from these.

Proposition 3.14. *Let $T = (L_1, L_2, \phi)$ be a holomorphic triple of type $(1, 1, d_1, d_2)$. If $\phi \neq 0$ then T is σ-stable if and only if $\sigma > d_1 - d_2 > 0$. If $\phi = 0$ then T is not σ-stable for any σ and it is semistable if and only if $\sigma = d_1 - d_2$.*

Proof. There are ony two candidates to non-trivial subtriples, namely $T' = (L_1, 0, 0)$ and $T'' = (0, L_2, 0)$. T' is indeed a subtriple which can be checked immediately. However, T'' does not give rise to a commutative diagram unless $\phi = 0$. Let $\sigma \in \mathbb{R}$. We will check what precise conditions need to be met for the triple (L_1, L_2, ϕ) to be σ-stable. Calculating σ-slopes yields:

$$\begin{aligned} \mu_\sigma(T) &= \frac{\deg(L_1 \oplus L_2) + 1 \cdot \sigma}{2} = \frac{d_1 + d_2 + \sigma}{2}, \\ \mu_\sigma(T') &= \frac{\deg(L_1 \oplus 0) + 0 \cdot \sigma}{1} = d_1, \\ \mu_\sigma(T'') &= \frac{\deg(0 \oplus L_2) + 1 \cdot \sigma}{1} = d_2 + \sigma. \end{aligned}$$

If $\phi \neq 0$, the only stability condition is $\mu_\sigma(T') < \mu_\sigma(T)$, or simply

$$d_1 - d_2 < \sigma.$$

In the case when $\phi \equiv 0$, then we get a contradiction

$$d_1 - d_2 < \sigma < d_1 - d_2,$$

from which we conclude that $(E_1, E_2, 0)$ is not a stable triple, and (E_1, E_2, ϕ) is a σ-stable triple if and only if $\sigma > d_1 - d_2$ (for $\phi \neq 0$). Note that in order for ϕ not to be trivial it is necessary that $d_1 - d_2 > 0$ as $\mathrm{Hom}(L_2, L_1) \simeq \mathcal{O}(d_1 - d_2)$ has no nontrivial global sections unless $d_1 - d_2 > 0$. We can study semistability by weakinging the inequalities. In this case, $(E_1, E_2, 0)$ is a σ-semistable triple if and only if $\sigma = d_1 - d_2$.

□

The moduli space of σ-stable triples of type $(1, 1, d_1, d_2)$ has been characterized in [31], yielding a projective space.

Proposition 3.15 (Corollary 3.2.1 in [31]). *Let $\mathcal{M}^s_\sigma$ be the moduli space of stable holomorphic triples of type $(1, 1, d_1, d_2)$ with $d_1 \geq d_2$. Then for $\sigma > d_1 - d_2$,*

$$\mathcal{M}^s_\sigma \simeq \mathrm{Sym}^{d_1 - d_2}(\mathbb{P}^1) = \mathbb{P}^{d_1 - d_2}.$$

The above expression of the moduli space of stable triples of type $(1, 1, d_1, d_2)$ hints at the action of $SL(2, \mathbb{C})$. Recall that $SL(2, \mathbb{C})$ acts naturally on $\mathbb{P}^1$, and therefore an action on the symmetric product is available

$$g \cdot (p_1 \odot \cdots \odot p_{d_1 - d_2}) := (g \cdot p_1) \odot \cdots \odot (g \cdot p_{d_1 - d_2}).$$

Also note that the d-fold symmetric product of $\mathbb{P}^1$ is in one-to-one correspondence with the set of degree d effective divisors on $\mathbb{P}^1$, an interpretation that ties the relationship between stable holomorphic triples and the Abelian gravitational vortex equations.

Let us now turn to higher ranks. Consider triples of the form (L_1, E_2, ϕ), where L_1 is a line bundle and E_2 is a rank-two vector bundle. By the classification of holomorphic bundles over $\mathbb{P}^1$ we have to study all possible holomorphic maps

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \xrightarrow{\phi} \mathcal{O}(c)$$

for integers a, b, c. The problem of classifying stable holomorphic triples with ranks $r_1 = 1, r_2 = 2$ has been partially undertaken in [31], but only for specific degrees, namely $d_1 = 0$ (i.e. $L_1 = \mathcal{O}$) and $\deg E_2 = -s$. Let us restrict then to holomorphic triples of the form

$$T : \mathcal{O}(-d) \oplus \mathcal{O}(-e) \xrightarrow{\phi} \mathcal{O},$$

with $d + e = s$ and let us assume without loss of generality $e \geq d$. We can show that T can only be stable if $s > 1$.

Proposition 3.16. *If T is σ-stable, then $d > 0$. In particular $s = d + e \geq 2d > 1$.*

Proof. By contradiction. Assume $d < 0$ Then $T' = (0, \mathcal{O}(-d), 0)$ is a subtriple of T. However, it is also a quotient triple induced by the subtriple $(\mathcal{O}, \mathcal{O}(-e), \phi|_{\mathcal{O}(-e)})$. Therefore by σ-stability of T we get that simultaneously $\mu_\sigma(T') < \mu_\sigma(T)$ and $\mu_\sigma(T') > \mu_\sigma(T)$, a contradiction.

Assume now that $d = 0$. The triple is then given by $T = (\mathcal{O}, \mathcal{O} \oplus \mathcal{O}(-s), \phi)$. Consider the case in which $\phi(\mathcal{O}) = 0$. Then $T'' = (0, \mathcal{O}, 0)$ is both a subtriple and a quotient triple (induced by the subtriple $(\mathcal{O}, \mathcal{O}(-s), \phi|_{\mathcal{O}(-s)})$), and we get another contradiction. If $\phi(\mathcal{O}) \neq 0$, then ϕ is surjective and $\ker \phi = \mathcal{O}(-s)$, and the subtriple $(\mathcal{O}, \mathcal{O}, \phi|_{\mathcal{O}})$ is a quotient triple induced by $(0, \mathcal{O}(-s), 0)$, leading again to a contradiction with the stability of T. □

Therefore the analysis is restricted to holomorphic triples of type $(1, 2, 0, -s)$ for $s > 1$. The interval for σ for triples of this type to be σ-stable is bounded and given by the following result, which also specifies the finite number of points where stability fails.

Proposition 3.17 (Lemma 4.1.1 and Proposition 4.1.3 in [31])**.** *Let $T = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e), \phi)$ be a holomorphic triple of type $(1, 2, 0, -s)$, with $0 < d \leq e$. If T is σ-stable, then*

$$\sigma \in (\sigma_0, 2s),$$

where $\sigma_0 = \frac{s}{2}$ for s even and $\sigma_0 = \frac{s+3}{2}$ for s odd. Furthermore, the subintervals in which the stability conditions is independent of σ are given by the decomposition

$$(\sigma_0, 2s) = (\sigma_0, \sigma_0 + 3) \sqcup (\sigma_0 + 3, \sigma_0 + 6) \sqcup \cdots \sqcup (2s - 6, 2s - 3) \sqcup (2s - 3, 2s).$$

For the extremal (leftmost and rightmost) subintervals, the moduli space of σ-stable triples of type $(1, 2, 0, -s)$ has been precisely computed. We recall these results in the following proposition.

Proposition 3.18. *If $\sigma \in (2s-3, 2s)$, the moduli space of holomorphic triples of type $(1, 2, 0, -s)$ is given by*

$$\mathcal{M}^s_\sigma(1, 2, 0 - s) \simeq \mathbb{P}^{s-2}.$$

If s is even and $\sigma \in (\alpha_0, \alpha_0 + 3)$, the moduli space is given by

$$\mathcal{M}^s_\sigma(1, 2, 0, -s) \simeq Gr\left(2, \frac{s}{2} + 1\right).$$

3.6 Further research

Explicit descriptions of the moduli spaces of holomorphic triples will play an important role in the study of our proposed Conjecture 3.12. A possible course of action can be outlined: firstly, it seems important to describe further moduli spaces of stable triples for arbitrary ranks and degrees. Secondly, and supported on previous descriptions, a explicit construction of the action of $SL(2,\mathbb{C})$ on this moduli space is prerequisite to studying GIT stability on this space.

Based on the nature of previous results we expect that one direction of the conjecture will be easier to prove, namely that the existence of a solution to the gravitational vortex equations implies the GIT polystability of the holomorphic triple; the converse will probably be much harder.

Another possible research direction could be to consider the Kähler–Yang–Mills–Higgs equations. Introduced by Álvarez-Cónsul and García-Prada in [3], these equations generalize the KYM equations by allowing the Higgs field ϕ to vary. Several techniques have been developed and explicit obstructions in the style of Futaki invariants have been constructed. It is promising to consider these obstructions to find negative answers on the existence of solutions to the gravitational vortex equations on general Riemann surfaces. The authors in [3] also consider general dimensional reductions through the action of semisimple complex Lie groups $K^{\mathbb{C}}$ and parabolic subgroups $P \subset K^{\mathbb{C}}$, and they prove a correspondence between K-invariant solutions to the KYM equations and some type of vortex equations over quiver bundles, which consist of a collection of vector bundles and morphisms between them. In the particular case of the $SU(2)$-invariant KYM equations, the authors obtain a necessary condition for the existence of solutions involving the degrees of the vector bundles and the number of zeros of the Higgs field.

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